Numerical analysis of a static cylindrically symmetric Abelian Higgs sunspot

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Abstract. We present the exact numerical solutions of the Ginzburg-Landau equations for the case of a static, cylindrically symmetric Abelian Higgs sunspot model. The method of solving of these equations is presented in detail, and the behaviour of the Higgs field amplitude, magnetic field strength, electric current density, as well as of the diagonal components of the stress-energy tensor are illustrated for spots carrying one to five flux quanta.

Key words: Sun: sunspots

1. Introduction

Saniga (1990, 1992) pointed out that there exists a remarkable similarity between the structure of the standard *sunspot* and that of the *quantized magnetic vortex* in a type II superconductor. Namely, it has been demonstrated that the fundamental *geometric* and *(electro)magnetic* properties of an isolated, fairly symmetric spot can be reproduced well by the *classical* vortex solution of the Ginzburg-Landau type Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} + \frac{1}{2}\left(\frac{\partial\Phi}{\partial x^{\iota}} + igA_{\iota}\Phi\right)\left(\frac{\partial\Phi^{\star}}{\partial x^{\rho}} - igA_{\rho}\Phi^{\star}\right)\eta^{\iota\rho} - -\frac{\lambda}{4}\left(\Phi\Phi^{\star} - \frac{m^2}{\lambda}\right)^2, \quad (1)$$

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which describes a locally gauge invariant theory of a complex scalar Higgs field Φ interacting with the electromagnetic gauge potential A_ϱ (henceforth the notation of Saniga (1990) is used). The aim of this article is to present in detail the method of (numerical) solution of the equations of motion following from (1) for the case of a static cylindrically symmetric sunspot, and show explicitly the radial distribution of the most important physical quantities inside the latter.

2. Formulation of the problem

Using a polar system of coordinates - r, φ, z - the cylindrically symmetric sunspot may be parametrized by the ansätz:

$$A_0 = 0 \ , \boldsymbol{A} = A(r)\hat{e}_\varphi \ , \ \Phi = f(r)exp(ip\varphi) \ , \qquad (2)$$

where $\boldsymbol{A}$ is the spatial part of the electromagnetic vector potential and p - an *integer* - stands for the number of flux quanta carried by the spot (Saniga 1990, 1992). Using dimensionless, reduced quantities the following equations of the Ginzburg-Landau type can be obtained from the Lagrangian density (1)(Saniga 1990):

$$-\frac{1}{K^2}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + f\left[\left(\frac{p}{Kr} - A\right)^2 + f^2 - 1\right] = 0, \qquad (3)$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA)\right] + \left(\frac{p}{Kr} - A\right)f^2 = 0. \qquad (4)$$

The important quantity K is, apart from a constant factor, identified with the ratio between the radii of the penumbra and umbra, $\sqrt{2}K = r_p/r_u$.

In order to make the system of ordinary differential equations (ODEs) (3)-(4) complete one has to define boundary conditions. It is obvious that the following conditions must be met in the center of any sunspot:

$$A(r = 0) = 0, \qquad (5)$$

$$f(r = 0) = 0. \qquad (6)$$

These can easily be verified by expanding, in the system of ODEs (3)-(4), both $A(r)$ and $f(r)$ into a Taylor series in the neighbourhood of the spot's center ($r = 0$). The other two boundary conditions are defined at spatial infinity. We require

$$f(r \to \infty) = 1 \qquad (7)$$

which physically corresponds to the nonzero value of the Higgs field vacuum. The last boundary condition can be obtained from equation (4) with the help of equation (7) :

$$A(r \to \infty) = 0, \qquad (8)$$

which is plausible from the physical point of view as it tells us that the influence of the flux tube (i.e, sunspot) vanishes at spatial infinity.

Equations (3)-(8) form a complete system.

3. Method of numerical solution

The system of ODEs (3)-(4) has been solved numerically by a relaxation method. The ODEs (3)-(4) have been rewritten to read

$$-\frac{1}{K^2}\left(f''+\frac{1}{r}f'\right)+f\left[\left(\frac{p}{Kr}-A\right)^2+f^2-1\right]=0 \tag{9}$$

and

$$A''+\frac{1}{r}A'-\frac{1}{r^2}A+\left(\frac{p}{Kr}-A\right)f^2=0, \tag{10}$$

where the prime denotes differentiation with respect to r.

One first replaces ODEs (9)-(10) by approximate finite difference equations (FDEs) on a grid of points that spans the domain $r \in< 0,\infty)$. We use the following standard approximations (see any textbook on numerical methods, e. g., Riečanová *et al.* 1987) :

$$\alpha'_j=\frac{\alpha_{j+1}-\alpha_{j-1}}{2h} \tag{11}$$

and

$$\alpha''_j=\frac{\alpha_{j+1}-2\alpha_j+\alpha_{j-1}}{h^2}, \tag{12}$$

where

$$\alpha_j=\begin{cases} f(r_j) \\ A(r_j) \end{cases} \tag{13}$$

and analogously for α' and α'' ; also

$$r_j=(j-1)h. \tag{14}$$

One obtains, after having substituted equations (11)-(14) into equations (9)-(10), the following relations for the j-th grid point:

$$f_j=\frac{1}{2(j-1)}\frac{(2j-1)f_{j+1}+(2j-3)f_{j-1}}{[p/K-(j-1)hA_j]^2/(j-1)^2+h^2(f_j^2-1)+2/K^2}, \tag{15}$$

$$A_j=\frac{1}{2(j-1)}\frac{(2j-1)A_{j+1}+(2j-3)A_{j-1}+2(p/K)hf_j^2}{2+h^2f_j^2+1/(j-1)^2}. \tag{16}$$

Equations (15)-(16) can only be used for $1<j<N$. The case $j=1$ corresponds to the center of the sunspot (see equation (14)), and boundary conditions (5)-(6) yield

$$f_1=0\ ,\ A_1=0. \tag{17}$$

The case $j=N$ corresponds to $r\to\infty$ (spatial infinity is replaced here by the value $r=(N-1)H$, see equation (14)). Boundary conditions (7)-(8) lead to

$$f_N=1\ ,\ A_N=0. \tag{18}$$

The relaxation method determines the solution by starting with a guess (which can be obtained, for example, by a variational procedure), and then using an iteration procedure. The iteration procedure is determined by the system of equations (15)-(16) for $1 < j < N$

$$A_j^n = \frac{(2j-1)\,A_{j+1}^o + (2j-3)\,A_{j-1}^o + 2\,(p/K)\,h\,\left(f_j^o\right)^2}{2 + h^2\left(f_j^o\right)^2 + 1/\left(j-1\right)^2}\,\frac{1}{2\,(j-1)} \tag{19}$$

and

$$f_j^n = \frac{(2j-1)\,f_{j+1}^o + (2j-3)\,f_{j-1}^o}{\left[p/K - (j-1)\,hA_j^n\right]^2/\left(j-1\right)^2 + h^2\left[\left(f_j^n\right)^2 - 1\right] + 2/K^2} \times$$
$$\times \frac{1}{2\,(j-1)}, \tag{20}$$

where the upper symbols 'o' and 'n' stand for the 'old' and 'new' (i.e., improved) value, respectively. Equations (19)-(20) are solved iteratively for $1 < j < N$ taking into account fixed boundary conditions (17)-(18). The first rough approximate solutions are refined by an iteration procedure until the following condition for relative errors of fields $A(r)$ and $f(r)$

$$\max_j \left(\left|1 - A_j^o/A_j^n\right| \;,\; \left|1 - f_j^o/f_j^n\right| \right) \;<\; \epsilon \;, \tag{21}$$

is obtained, ϵ representing the required accuracy.

Equation (20), as it stands, represents a cubic equation in the unknown quantity f_j^n, as $(f_j^n)^2$ appears in the denominator of this equation. This cubic equation has been solved by an iteration procedure, too. This is defined by the same equation (20). The value of f_j^n on the RHS of equation (20) is taken as the value obtained from the last iteration, and the improved approximation is defined by the LHS of the same equation; the value f_j^o has been chosen as a starting guess.

The last problem which one faces concerns the replacement of spatial infinity by the finite value $r_N = (N-1)h$ (see the equation (14) for $j = N$). Natural number N is always chosen in such way that even a several times larger value of the latter leads, within the required accuracy, to the same results for the computed physical quantities.

4. Results

Once we have succeeded in solving the ODEs (3)-(4), complemented by the boundary conditions (5)-(8), one can find some other important physical quantities as the functions of a radial distance from the center of a cylindrically symmetric sunspot.

The magnetic field strength vector $\boldsymbol{H}$ is, in general, defined by the formula

$$\boldsymbol{H} = \nabla \times \boldsymbol{A} \equiv curl \boldsymbol{A}. \tag{22}$$

Using cylindrical coordinates and ansatz (2) one obtains

$$H = \frac{1}{r}\frac{d}{dr}(rA), \tag{23}$$

where H represents the z- (the only non-zero) component of the magnetic field. We thus have

$$H = A' + \frac{1}{r}A \tag{24}$$

for $r > 0$, and

$$H = 2A' \tag{25}$$

for $r = 0$. It can easily be proved that $\lim_{r\to 0}(A/r) = H/2$, in very much the same way as relations (5)-(6) were obtained. A finite difference approximation of type (11) can be used to examine equation (24) (as well as equation (14)). Equation (25) is replaced by the following approximate finite difference equation:

$$H(r=0) \equiv H_1 = \frac{4A_2 - A_3}{h}, \tag{26}$$

if boundary condition (17) is considered (see, e. g., Riečanová *et al.* 1987).

One can further verify that in the case of cylindrically symmetric spot (2) the only non-zero components of the stress-energy tensor $T_{\varrho\kappa}$ are as follows

$$T_{00} = \frac{1}{2K^2}\left(\frac{df}{dr}\right)^2 + \frac{1}{2}f^2\left(\frac{p}{Kr} - A\right)^2 + \frac{1}{4}\left(f^2 - 1\right)^2 + \frac{1}{2}H^2, \tag{27}$$

$$T_{rr} = \frac{1}{2K^2}\left(\frac{df}{dr}\right)^2 - \frac{1}{2}f^2\left(\frac{p}{Kr} - A\right)^2 - \frac{1}{4}\left(f^2 - 1\right)^2 + \frac{1}{2}H^2 \tag{28}$$

and

$$T_{\varphi\varphi} = -\frac{r^2}{2K^2}\left(\frac{df}{dr}\right)^2 + \frac{r^2}{2}f^2\left(\frac{p}{Kr} - A\right)^2 - \frac{r^2}{4}\left(f^2 - 1\right)^2 + \frac{r^2}{2}H^2. \tag{29}$$

There is no difficulty in transforming the RHS of the last three equations into finite difference formulas for $r > 0$, using equations (11) and (13)-(14). A little more complicated is the case of the sunspot's center; the finite difference approximations of equations (27)-(29) then read (they can be derived in a way completely analogous to equation (26)):

$$T_{00}(r=0) = \frac{1}{2K^2}\left(1 + \delta_{1p}\right)\left(\frac{4f_2 - f_3}{2h}\right)^2 + \frac{1}{4} + \frac{1}{2}\left(H_1\right)^2, \tag{30}$$

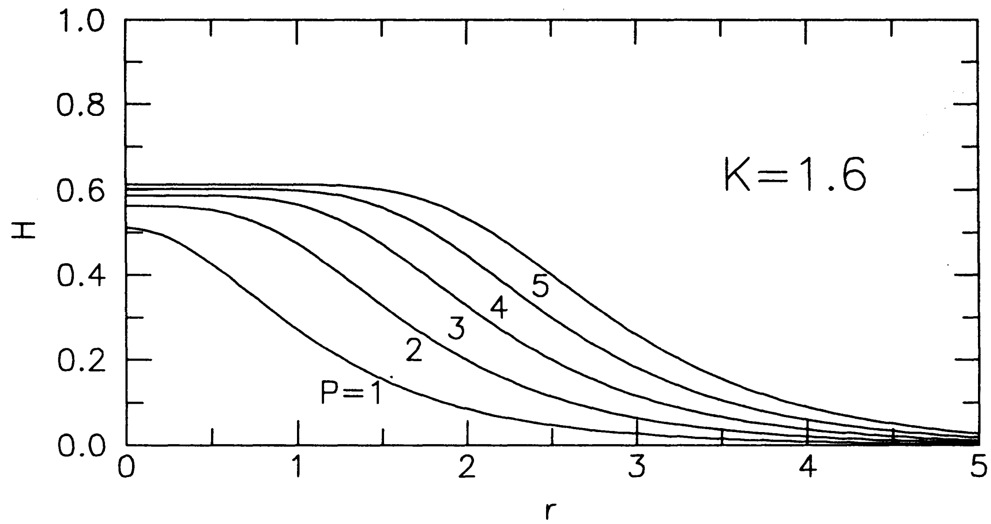

Figure 1. Magnetic field strength (H) distribution within a static cylindrically symmetric sunspot carrying p ($1 \leq p \leq 5$) magnetic flux quanta, plotted versus the radial distance (r) from the spot's center for ratio $K = 1.6$.

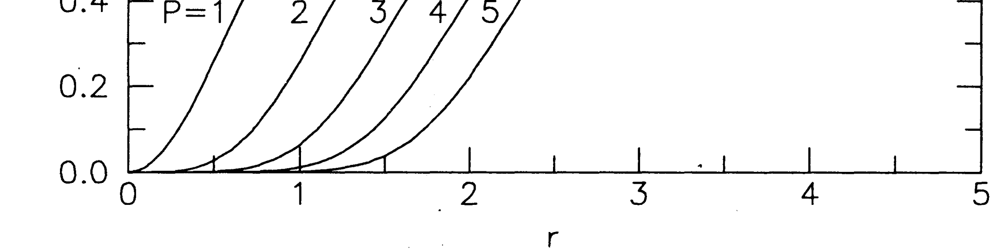

Figure 2. The same as in the Figure 1 for squared amplitude of the Higgs field (f^2).

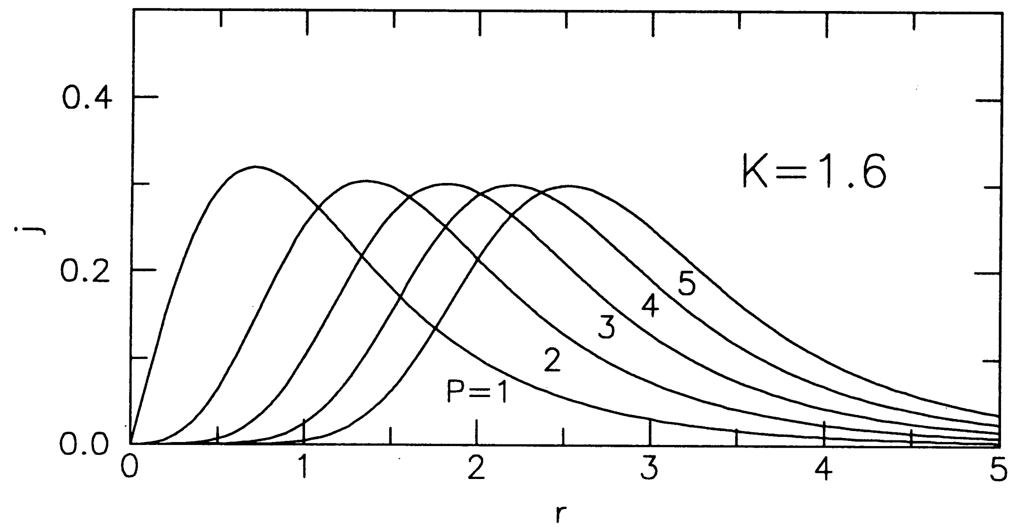

Figure 3. The same as in the Figure 1 for the φ-component of the electric current density vector.

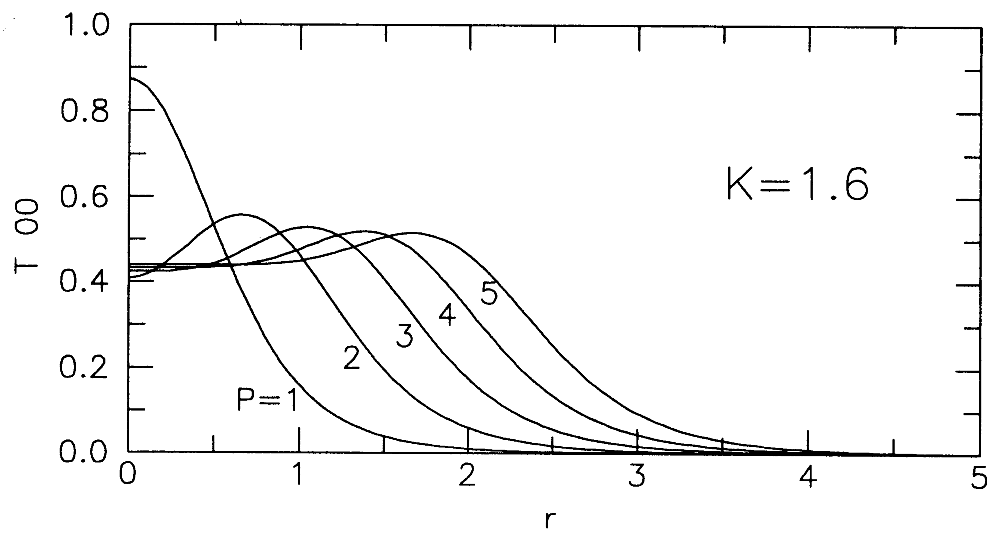

Figure 4. The same as in the Figure 1 for the energy density.

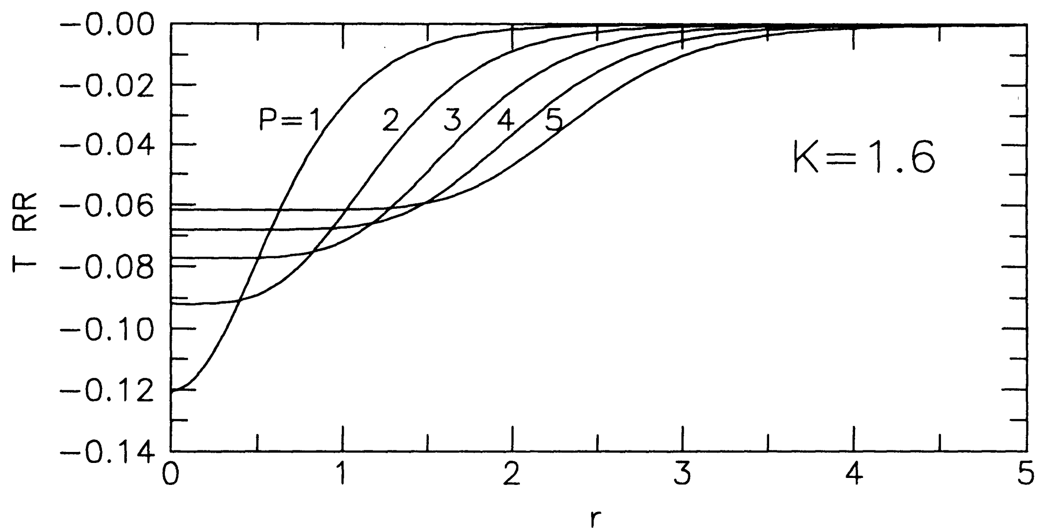

Figure 5. The same as in the Figure 1 for the rr-component of the stress-energy tensor.

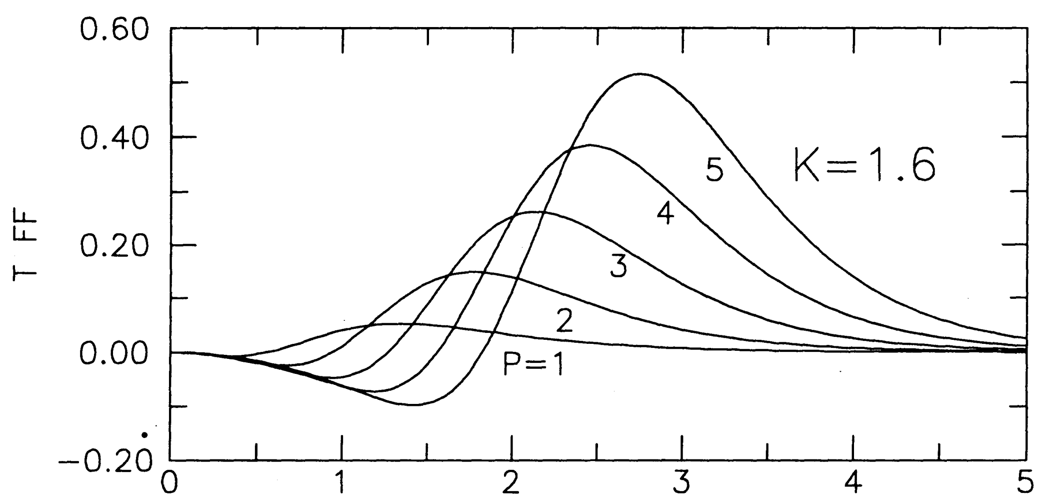

Figure 6. The same as in the Figure 1 for the $\varphi\varphi$-component of the stress-energy tensor.

$$T_{rr}(r=0) = \frac{1}{2K^2}(1-\delta_{1p})\left(\frac{4f_2 - f_3}{2h}\right)^2 - \frac{1}{4} + \frac{1}{2}(H_1)^2 , \tag{31}$$

and

$$T_{\varphi\varphi}(r=0) = 0, \tag{32}$$

where $\delta_{1p} = 1$ if $p = 1$ and zero otherwise.

Finally, the (only) nontrivial component of the current density is given by formula (Saniga 1990)

$$j_\varphi \equiv \boldsymbol{j} \cdot \hat{e}_\varphi = f^2 \left(\frac{p}{Kr} - A\right), \tag{33}$$

which vanishes in the center of sunspot. Since the purpose was to determine the value of the magnetic field strength in the spot's center (H_1) which deviated no more than $\approx 1\%$ from the 'real' value, one was forced to take the step $h = 0.05$ and $r_N = 10$ to approximate 'spatial infinity' for $\epsilon = 10^{-6}$. Some of our theoretical findings are visualised in the figures 1 to 6 (as in the previous sections we use dimensionless, reduced quantities as introduced in Saniga 1990).

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