Optical air mass and refraction in a Rayleigh atmosphere

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Abstract. Analytical expressions for the spectral optical air mass m and the angle of refraction ω in a molecular Rayleigh atmosphere were found in dependence on the air density profile. The correlation between m and ω is discussed. The theoretical expressions obtained in this paper show that both the optical air mass and the refraction angle close to the horizon depend critically on the vertical profile of the air density and may vary over a wide range ($\approx 35 - 43$' for m and $\approx 31 - 44$' for ω at the Earth's surface). This fact implies that the formulae independent of any meteorological parameter cannot express the optical air mass or the refraction angle with a sufficient accuracy. Therefore, comparing such formulae on the base of their accuracy has no sense.

Key words: optical air mass – angle of refraction – Rayleigh atmosphere

1. Introduction

The processes in the Earth's atmosphere induced by solar radiation strongly depend on the transmission and transformation of direct solar radiation. The attenuation of direct solar radiation is described well by Bouger's law, which has been repeatedly confirmed experimentally in a cloudless atmosphere in the visible and infrared spectrum (reference in Gushchin and Vinogradova, 1983).

If $I_{0\lambda}$ denotes the extraterrestrial flux density of direct solar radiation, then Bouger's law may be expressed in the form:

$$I_\lambda(\xi_0, h_0) = I_{0\lambda}\, exp\left\{-\int_{h_0}^{\infty} \alpha_{ext,\lambda}(h)\, sec\, \xi_\lambda(h)\, dh\right\}, \qquad (1)$$

where $I_\lambda(h_0)$ is the flux density of direct solar radiation at altitude h_0, $\alpha_{ext,\lambda}(h)$ the atmospheric spectral volume extinction coefficient, $\xi_\lambda(h)$ the apparent zenith angle of the Sun at altitude h, and ξ_0 the apparent zenith angle of the Sun at altitude h_0.

The relative optical air mass $m_{rel,\lambda}(\xi_0, h_0)$ expressed as

$$m_{rel,\lambda}(\xi_0, h_0) = \frac{\int_{h_0}^{\infty} \alpha_{ext,\lambda}(h)\, sec\, \xi_\lambda(h)\, dh}{\int_{h_0}^{\infty} \alpha_{ext,\lambda}(h)\, dh}. \qquad (2)$$

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is often taken into account in meteorological and astronomical applications to describe the transmission of radiation in the Earth's atmosphere. When different substances contributing to attenuation have different vertical distributions, each has its own dependence of air mass m on solar zenith angle. Many applications of evaluating the attenuation of direct solar radiation are based on the Langley plot method (Thomason and Herman, 1983). This method assumes that the relative concentrations of the attenuating substances are all identically distributed along the vertical. However, it is generally not possible to determine the mean air-mass values for water vapour and aerosols, due to their highly variable distributions (Thomason and Herman, 1983; Russel et al., 1993).

Many tables and approximations for air mass have so far been produced (Gushchin and Vinogradova, 1983; Baur, 1953; Mc Cartney, 1979; Kasten and Young, 1989; Kondrat'jev, 1954; Machotkin, 1960; Piršel, 1991; Swider and Gardner, 1967; Young, 1994). New revised tables and approximation formulae for optical air mass were published by Kasten and Young (1989). The analytical expressions for optical air mass are frequently necessary (e.g., for fast numerical calculations to solve the inverse and direct problems of atmospheric optics). For example, Gushchin and Vinogradova (1983) and Machotkin (1960) expressed $m(\xi_0)$ using a simple transcendental function. On the other hand, some empirically obtained relations for m (for example Piršel, 1991) are not suitable for application since they have not a physical basis (Kasten, 1993). The optical air mass is often expressed using the Chapman function, which is physically well-founded and has a suitable accuracy (Swider and Gardner, 1967). Other expressions for the optical air mass have been derived; e.g., the relationship between m and the angle of refraction ω (Mc.Cartney, 1979).

In the molecular atmosphere expression (1) has the form

$$I_\lambda(\xi_0, h_0) = I_{0\lambda}\; e^{-m_{rel,\lambda}(\xi_0,h_0).\tau_{m,\lambda}(h_0)} \ , \qquad (3)$$

where $\tau_{m,\lambda}(h_0)$ is the optical thickness of the molecular atmosphere and $m_\lambda(\xi_0, h_0)$ the relative optical air mass at altitude h_0

The analytical expressions for optical air mass $m_\lambda(\xi_0, h_0)$ and angle of refraction $\omega_\lambda(\xi_0, h_0)$ in a uniformly mixed Rayleigh atmosphere (presented here) were found by solving the following integrals,

$$m_{abs,\lambda}(\xi_0, h_0) = \int_{h_0}^{\infty} \rho(h)\; sec\; \xi_\lambda(h)\; dh \ , \qquad (4)$$

$$\omega_\lambda(\xi_0, h_0) = \int_{h_0}^{\infty} d\xi_\lambda(h) = \int_{h_0}^{\infty} \frac{d\; sin\; \xi_\lambda(h)}{dh}\; sec\; \xi_\lambda(h)\; dh, \qquad (5)$$

where $m_{abs,\lambda}$ is the absolute optical air mass. The volume extinction coefficient for the molecular atmosphere is proportionate to air density ρ. We have used ρ instead of α_{ext} to simplify the following derivations.

The resulting expressions can be applied for fast and accurate numerical calculations due to their analytical form. There is a point to be stressed. Different authors prefer different expressions for optical air mass. Some proposed formulae are independent of any meteorological parameter, but their preference is often based on accuracy of approximation of real values (in order of some percent). However, it will be shown in the following paragraphs that the theoretical values of m in a pure molecular atmosphere may vary over a wider range than a few percent due to varying air density. It is about ten percent for $\xi_0 > 80^o$.

2. Optical air mass

Integral (4) can be expressed as

$$m_{abs,\lambda}(\xi_0, h_0) = \int_{h_0}^{\infty} \frac{\rho(h)\,(R_E + h)\, n_\lambda(h)}{\sqrt{(R_E+h)^2 n_\lambda^2(h) - (R_E + h_0)^2 n_\lambda^2(h_0)\, sin^2\xi_0}}\, dh\ , \quad (6)$$

when we apply Snell's law in the spherical atmosphere. The Earth's radius is denoted by R_E and $n_\lambda(h)$ is the refractive index of the atmosphere at altitude h.

The vertical structure of the air density can be highly sensitive to the p-resence of inversion and other lapse rate. However, one can show that also for a simply exponential form of density as a function of height, the values of the refraction angle and optical air mass near the horizon may vary over a wide range. The refractive index can be expressed as follows:

$$n_\lambda(h) = 1 + c_\lambda\, \rho(h)\ , \quad (7)$$

where

$$\rho(h) = \rho_0 . e^{-\beta h}\ , \quad (8)$$

c_λ is the light dispersion coefficient (Link, 1956) and $\beta = 1/H$ where H is the scale height. The value of $(n_\lambda - 1)$ is much more smaller than 1 in the visible and infrared spectrum. We then obtain the optical air mass

$$m_{abs,\lambda}(\xi_0, h_0) = \frac{\rho_0}{2\beta}\, exp\left\{\beta\left(F_\lambda(\xi_0, h_0) - h_0\right)\right\} \sqrt{\frac{\pi\beta R_E}{2}} \left[1 - erf\left(\sqrt{\beta F_\lambda(\xi_0, h_0)}\,\right)\right] \cdot$$

$$\cdot \left[2 + \frac{1}{\beta R_E} - \frac{2}{R_E}\left(F_\lambda(\xi_0, h_0) - h_0\right)\right] + \frac{\rho_0}{2\beta}\, e^{-\beta h_0} \sqrt{\frac{2}{R_E}\, F_\lambda(\xi_0, h_0)}\ , \quad (9a)$$

where

$$F_\lambda(\xi_0, h_0) = \left(\frac{R_E}{2} + h_0\right)\, cos^2\xi_0 - R_E c_\lambda \rho(h_0)\, sin^2\xi_0 .$$

The expression (9a) can be overwritten using the relations $h_0/R_E \ll 1$ and $c_\lambda \rho(h_0) \ll 1$ as follows:

$$m_{abs,\lambda}(\xi_0, h_0 \approx 0) = \frac{\rho_0}{2\beta}\, exp\left\{\frac{\beta R_E}{2}\, cos^2\xi_0\right\} \sqrt{\frac{\pi\beta R_E}{2}} \left[1 - erf\left(\sqrt{\frac{\beta R_E}{2}}\, cos\xi_0\right)\right] \cdot$$

$$\cdot\left[1+sin^2\xi_0+\frac{1}{\beta R_E}\right]+\frac{\rho_0}{2\beta}\,cos\xi_0. \qquad (9b)$$

The limit values of absolute optical air mass in the zenith for both previous cases are given as follows:

$$m_{abs,\lambda}(\xi_0=0,h_0)=\frac{\rho_0}{2\beta}\,e^{-\beta h_0}\left(2+\frac{1}{\beta R_E}\right)\approx\frac{\rho_0}{\beta}\,e^{-\beta h_0} \qquad (10a)$$

$$m_{abs,\lambda}(\xi_0=0,h_0=0)=\frac{\rho_0}{2\beta}\left(2+\frac{1}{\beta R_E}\right)\approx\frac{\rho_0}{\beta} \qquad (10b)$$

It can be shown, using the data published by Link and Neužil (1965), that m_{abs} may vary from 7500 to 9800 $kg.m^{-2}$ in dependence on the geographical coordinates and meteorological conditions. The relative optical air mass calculated according to Eqs. (9a-b) and using (10a-b) is then:

$$m_{rel,\lambda}(\xi_0,h_0)=\frac{1}{2}\,e^{\beta F_\lambda(\xi_0,h_0)}\sqrt{\frac{\pi\beta R_E}{2}}\left[1-erf\left(\sqrt{\beta F_\lambda(\xi_0,h_0)}\right)\right]\cdot$$

$$\cdot\left[2+\frac{1}{\beta R_E}-\frac{2}{R_E}(F_\lambda(\xi_0,h_0)-h_0)\right]+\frac{1}{2}\sqrt{\frac{2}{R_E}\,F_\lambda(\xi_0,h_0)} \qquad (11a)$$

or

$$m_{rel,\lambda}(\xi_0,h_0\approx 0)=\frac{1}{2}\,exp\left\{\frac{\beta R_E}{2}\,cos^2\xi_0\right\}\sqrt{\frac{\pi\beta R_E}{2}}\left[1-erf\left(\sqrt{\frac{\beta R_E}{2}}\,cos\xi_0\right)\right]\cdot$$

$$\cdot\left[1+sin^2\xi_0+\frac{1}{\beta R_E}\right]+\frac{cos\xi_0}{2}. \qquad (11b)$$

Expression (11b) is very similar to the well-known Chapman function. The results of calculation (according to Eq.11b) are presented in Tab.1.

3. Refraction

Refraction integral (5) may be modified to read (similar as m in Section 2):

$$\omega_\lambda(\xi_0,h_0)=\beta c_\lambda\rho_0(R_E+h_0)\,sin\,\xi_0\cdot$$

$$\cdot\int_{h_0}^{\infty}\frac{e^{-\beta h}\;n_\lambda(h)}{\sqrt{(R_E+h)^2n_\lambda^2(h)-(R_E+h_0)^2n_\lambda^2(h_0)\,sin^2\xi_0}}\,dh\,. \qquad (12)$$

Integrating leads to the complex formula

$$\omega_\lambda(\xi_0,h_0)=c_\lambda\rho_0\sqrt{\frac{\beta(R_E+h_0)\,sin\,\xi_0}{2}}\,exp\left\{\beta R_E-\beta(R_E+h_0)\,sin\xi_0\right\}.$$

Table 1. Optical air mass calculated according Eq.(11).

$\xi_0[°]$	m_{min}	$m_{average}$	m_{max}
50.0	1.5534	1.5541	1.5544
55.0	1.7395	1.7406	1.7411
60.0	1.9930	1.9949	1.9957
65.0	2.3533	2.3566	2.3580
70.0	2.8973	2.9039	2.9067
72.0	3.1993	3.2083	3.2121
74.0	3.5753	3.5880	3.5933
76.0	4.0549	4.0736	4.0813
78.0	4.6857	4.7147	4.7266
80.0	5.5489	5.5968	5.6165
81.0	6.1096	6.1731	6.1993
82.0	6.7934	6.8790	6.9148
83.0	7.6422	7.7620	7.8121
84.0	8.7201	8.8931	8.9663
85.0	10.1262	10.3865	10.4982
86.0	12.0213	12.4324	12.6125
87.0	14.6792	15.3678	15.6784
87.5	16.4352	17.3499	17.7701
88.0	18.5950	19.8339	20.4158
88.5	21.2941	23.0089	23.8353
89.0	24.7272	27.1581	28.3655
89.5	29.1812	32.7186	34.5391
90.0	35.0876	40.3853	43.2279

$$. \sum_{n=0}^{\infty}(-1)^n \frac{1.3.5...(2n-1)}{4^n n!\,[\beta(R_E+h_0)\ sin\ \xi_0]^n}\ \Gamma\left(n+\frac{1}{2}, \beta(R_E+h_0).(1-sin\ \xi_0)\right)\ , \tag{13}$$

which is not suitable for numerical applications. $\Gamma(\alpha, x^2)$ in Eq. (13) is an incomplete gamma function, which can be expressed using the degenerate hyperbolic function $\phi(\alpha, 1+\alpha; x^2)$. However, it is advantageous to apply the following recurrent relation

$$\Gamma\left(n+\frac{1}{2}, x^2\right) = \left(n-\frac{1}{2}\right)\ \Gamma\left(n-\frac{1}{2}, x^2\right) + x^{2n-1}\ e^{-x^2}\ .$$

for numerical calculation. In the limiting case

$$\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi}\ [1-erf(x)]\ .$$

However, we may to express refraction angle ω also in the following simple form: form also:

$$\omega_\lambda(\xi_0, h_0) = c_\lambda \rho_0 (R_E+h_0)\ sin\ \xi_0\ \sqrt{\frac{\pi\beta}{2R_E}}\ e^{-\beta h_0}\ exp\left\{\left(\frac{R_E}{2}+h_0\right)\beta\ cos^2\xi_0\right\}$$

Table 2. Comparison of refraction angles (in ') calculated according Eqs.(13) and (14) for $h_0 = 0$ m.

$\xi_0[°]$	$\omega_{min}(12)$	$\omega_{min}(13)$	$\omega_{average}(13)$	$\omega_{average}(14)$	$\omega_{max}(13)$	$\omega_{max}(14)$
65.0	2.1741	2.1197	2.1777	2.1233	2.6686	2.6018
70.0	2.7473	2.7038	2.7541	2.7106	3.3754	3.3221
72.0	3.0593	3.0202	3.0686	3.0295	3.7612	3.7132
74.0	3.4438	3.4091	3.4568	3.4220	4.2375	4.1949
76.0	3.9294	3.8990	3.9482	3.9178	4.8408	4.8036
78.0	4.5618	4.5359	4.5905	4.5646	5.6300	5.5982
80.0	5.4181	5.3967	5.4648	5.4433	6.7052	6.6788
81.0	5.9703	5.9511	6.0310	6.0117	7.4022	7.3785
82.0	6.6385	6.6216	6.7201	6.7031	8.2512	8.2303
83.0	7.4620	7.4475	7.5745	7.5597	9.3051	9.2870
84.0	8.4985	8.4865	8.6582	8.6459	10.6441	10.6289
85.0	9.8355	9.8265	10.0711	10.0613	12.3935	12.3813
86.0	11.6092	11.6069	11.9733	11.9680	14.7557	14.7485
87.0	14.0558	14.0586	14.6412	14.6428	18.0807	18.0823
87.5	15.6486	15.6516	16.4085	16.4112	20.2911	20.2943
88.0	17.5812	17.5840	18.5854	18.5880	23.0226	23.0257
88.5	19.9572	19.9602	21.3087	21.3112	26.4524	26.4554
89.0	22.9226	22.9263	24.7769	24.7800	30.8397	30.8434
89.5	26.6857	26.6906	29.2840	29.2882	36.5711	36.5762
90.0	31.5423	31.5554	35.2683	35.2800	44.2291	44.2431

$$\cdot\left[1 - erf\left(\sqrt{\left(\frac{R_E}{2} + h_0\right)\beta}\, cos\, \xi_0\right)\right] . \tag{14}$$

The comparison of the values of ω calculated according Eqs. (13) and (14) is presented in Tab.2.

The calculation of the optical air mass presented in some papers was proposed from the known profile of refraction angle ω. For example, the relationship between m and ω was expressed by McCartney (1979) as

$$\omega_\lambda(\xi_0) = 58.36".m_\lambda(\xi_0)\; sin\; \xi_0 \; . \tag{15}$$

This behaves relatively well for $\xi_0 < 80^o$ in an average atmosphere (for a mean vertical profile of air density). But, the differences between the values of ω calculated according to Eq. 15 and the values obtained by regular integration are quite high near the horizon (Fig. 1). These differences are approximately 20% for large zenith angles.

4. Conclusion

The analytical formulae for optical air mass $m(\xi_0, h_0)$ and the angle of refraction $\omega(\xi_0, h_0)$ have been presented. Values of $m(\xi_0 = \frac{\pi}{2}, 0)$ and $\omega(\xi_0 = \frac{\pi}{2}, 0)$ strongly

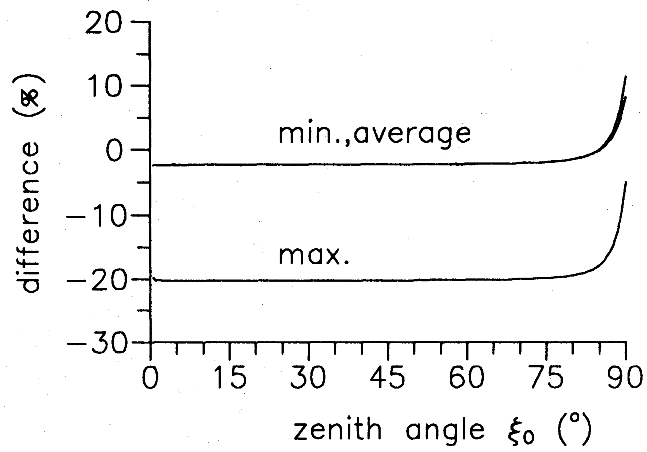

Figure 1. Deviation between the refraction angle calculated approximately (according Eq.15) and precisely (according Eq.(4)).

depend on meteorological conditions (in this paper characterized by the air density at the Earth's surface ρ_0 and by the vertical gradient of air density β) and may vary over a wide range ($\approx 35 - 43$ for m, and $\approx 31 - 44$' for ω at the Earth's surface). However many formulae were obtained empirically and have not a physical basis - are independent of any meteorological parameter. This fact implies that such formulae can not to express the optical air mass or refraction angle with sufficient accuracy. therefore, the comparison of accuracy of results obtained using these formulae has no ground.

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