Contrib. Astron. Obs. Skalnaté Pleso **33**, 179 – 192, (2003)

Applications of the theory of the general three-body problem

N.A. Solovaya

Astronomical Institute of the Slovak Academy of Sciences, Interplanetary Matter Division, Dúbravská cesta 9, 845 04 Bratislava, The Slovak Republic

Received: March 21, 2003

Abstract. The analytical theory of the general three-body problem to real triple stellar systems is applied. On the stellar systems ξ UMa, ζ Aqu, and AS Cam it is shown how this theory can be used for the investigation of the dynamical evolution of triple stellar systems. It is possible to establish the moment of the close approach of the inner pair, and to define the secular and long-periodic perturbations in the motions of the periastrons and nodes.

Key words: three-body problem – dynamical evolution – stability – close approach

1. Introduction

The applications of the analytical theory of the case of the general three-body problem to the real triple systems are presented. The theory is valid for systems whose components are point-masses and no other forces act on them except for Newton's law of gravitation.

The first application of the theory was presented on the triple stellar system of ε Lyrae type (Orlov and Solovaya, 1988). It was shown that the theory permits to investigate the evolutionary changes of components of triple stellar systems concerning the mutual inclination. In the present paper we offer new applications of the theory to triple stellar systems. It is shown that the theory allows to investigate the characteristics of the approaches of components and their periodicity, and to define the time interval when the periastron distance cannot exceed the limit in which the stellar systems remain stable.

As it is well known, two types of triple stellar systems are observed. The first type of the systems consists of components whose mutual separations are of the same order, the second type consists of a close binary pair and a third distant component. The used theory (Orlov and Solovaya, 1974) is valid for the second type of stellar systems.

The motion of a close pair and a distant component is under large mutual perturbations. Therefore it is interesting to know the dynamical evolution of such stellar systems on a large time interval. For this purpose as the intermediate orbit we used the solution of the simplified canonical system of differential equations (formulae (1) in Solovaya and Pittich, 2002), in which the Hamiltonian does not contain terms of the third and higher orders. The secular and long-periodic terms to the second order were taken into account in the intermediate orbit. The analytical formulae of the theory allow to calculate extremal values of the eccentricities, the motion of both periastrons, and nodes, and to consider the question of the stability on a cosmogonic time-scale.

In the paper the theory is illustrated on the three different cases of the stellar systems. In the first case the theory is applied to the stellar system ξ UMa for which all six Keplerian elements and the directions of radial velocities in the vicinity of the nodes are known. Such orbits of stars in the triple system are defined unambiguously. All dynamical characteristics of such a system can be calculated by the formulae of the theory.

In the second case the theory is applied to systems for which the inclinations of the orbits in stellar catalogues are given with double signs. The theory allows to get the precise sign of the inclinations, because for one of the signs the system is unstable. To check this result the real stellar system ζ Aqu was investigated.

The third application deals with close binary systems, supposing the existence of a far third component. For such a system all orbital elements and the mass of the third component cannot be obtained from observations. Only the orbital period and the light equation may be known for the third component. The theory allows to get orbital elements of the supposed component so that the close binary system should remain stable. For the illustration the system AS Cam is considered.

2. Solution of problem

The presented system of formulae is the solution of the one case of the general three-body problem. It can be used for the investigation of the dynamical evolution of triple stellar systems in which the masses of the components are comparable and the distance between two of them is much smaller than the distance of either from the third.

In the solution (Orlov and Solovaya, 1974) the secular and the long-periodic terms to the second order were taken into account in the intermediate orbit. The motion is considered in the Jacobian coordinate system in canonical Delaunay elements. The invariable plane is the reference plane. The canonical elements are:

$$\begin{aligned} L_i &= \beta_i \sqrt{a_i}\,, \qquad G_i = L_i\sqrt{1-e_i^2}\,, \qquad H_i = G_i \cos i_i\,, \\ l_i &= M_i\,, \qquad g_i = \omega_i\,, \qquad h_i = \Omega_i\,, \end{aligned} \tag{1}$$

where

$$\beta_1 = k\,\frac{m_0 m_1}{\sqrt{m_0+m_1}}, \qquad \beta_2 = k\,\frac{(m_0+m_1)\,m_2}{\sqrt{m_0+m_1+m_2}}\,. \tag{2}$$

In the previous expressions the notations have the usual meaning: k – the Gaussian constant, a_i – the semi-major axis, e_i – the eccentricity, i_i – the inclination, M_i – the mean anomaly, and w_i – the argument of the periastron. The subscript i is equal to 1 for the inner obit and to 2 for the outer one. The short-periodic terms were excluded by application of von Zeipel's method. The general solution of the simplified system of differential equations was obtained in terms of hyperelliptic integrals by the Hamilton-Jacobi's method in the form:

$$\begin{aligned} L_1 &= \beta_1 \sqrt{a_1} = A_1\,, \qquad & L_2 = \beta_2 \sqrt{a_2} = A_2\,, \\ G_1 &= L_1 \sqrt{\xi}\,, \qquad & G_2 = L_2 \sqrt{1 - e_2^2} = A_4\,, \end{aligned} \tag{3}$$

where $\xi = \xi_1 + (\xi_2 - \xi_1)\, \mathrm{sn}^2 u\,,$

$$l_1 = B_1 + \kappa_1 (t - t_0) + Q_1 I_1(u) + Q_2 I_2(u) + Q_3 I_3(u)\,, \tag{4}$$

$$l_2 = B_2 + \kappa_2 (t - t_0)\ , \tag{5}$$

$$\begin{aligned} \sin g_1 &= -\frac{\sqrt{\xi_2 - \xi_1}\,\sqrt{\xi\,(\xi_5 - \xi)}}{\sqrt{5\,(1-\xi)\left[\xi - \left(\overline{c} - \overline{G}_2\right)^2\right]\left[\left(\overline{c} + \overline{G}_2\right)^2 - \xi\right]}}\,\mathrm{cn}u\,, \\ \cos g_1 &= +\frac{2\sqrt{(\xi_2 - \xi_1)\,(\xi_3 - \xi_1)}\,\sqrt{\xi_4 - \xi}}{\sqrt{5\,(1-\xi)\left[\xi - \left(\overline{c} - \overline{G}_2\right)^2\right]\left[\left(\overline{c} + \overline{G}_2\right)^2 - \xi\right]}}\,\mathrm{sn}u\,\mathrm{dn}u\,, \end{aligned} \tag{6}$$

for $\overline{h} < 0$ and

$$\begin{aligned} \sin g_1 &= \pm\frac{\sqrt{\xi_3 - \xi_1}\,\sqrt{\xi\ (\xi_5 - \xi)}}{\sqrt{5\,(1-\xi)\left[\xi - \left(\overline{c} - \overline{G}_2\right)^2\right]\left[\left(\overline{c} + \overline{G}_2\right)^2 - \xi\right]}}\,\mathrm{dn}u\,, \\ \cos g_1 &= \mp\frac{2\,(\xi_2 - \xi_1)\,\sqrt{\xi_4 - \xi}}{\sqrt{5\,(1-\xi)\left[\xi - \left(\overline{c} - \overline{G}_2\right)^2\right]\left[\left(\overline{c} + \overline{G}_2\right)^2 - \xi\right]}}\,\mathrm{sn}u\,\mathrm{cn}u\,, \end{aligned} \tag{7}$$

for $\overline{h} > 0$,

$$g_2 = B_4 + \kappa_4 (t - t_0) + Q_4 I_1(u) + Q_5 I_2(u) + Q_6 I_4(u) + Q_7 I_5(u)\,, \tag{8}$$

$$h_1 = B_5 + Q_8 I_1(u) + Q_6 I_4(u) - Q_7 I_5(u)\,, \tag{9}$$

where

$$I_1(u) \;=\; \int_0^u \frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,, \tag{10}$$

$$I_2(u) \;=\; \int_0^u \mathrm{sn}^2 u\,\frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,, \tag{11}$$

$$I_3(u) \;=\; \int_0^u \frac{\mathrm{sn}^2 u}{1-b_1^2\,\mathrm{sn}^2 u}\,\frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,, \tag{12}$$

$$I_4(u) \;=\; \int_0^u \frac{\mathrm{sn}^2 u}{1-b_2^2\,\mathrm{sn}^2 u}\,\frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,, \tag{13}$$

$$I_5(u) \;=\; \int_0^u \frac{\mathrm{sn}^2 u}{1+b_3^2\,\mathrm{sn}^2 u}\,\frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,, \tag{14}$$

$$\sigma \;=\; 1-2\,\beta\,\varepsilon\,\mathrm{sn}^2 u+\varepsilon^2\,\mathrm{sn}^4 u\,. \tag{15}$$

ξ_j $(j = 1,2,\ldots,5)$ are the roots of the polynomials of the 5-th order of the denominator in the obtained solution. The maximum and the minimum meaning of the eccentricity will be defined by formulae:

$$e_{1\,min} = \sqrt{1-\xi_2}, \qquad e_{1\,max} = \sqrt{1-\xi_1}\,, \tag{16}$$

where ξ_1 and ξ_2 are two roots of polynomials, which are less than 1. The coefficients Q_l $(l = 1,2,\ldots,8)$ depend on ξ_j, $\overline{c}$ and $\overline{G}_2$. They are obtained after differentiation, when we get the solution by Hamilton–Jacobi's method (formulas (XII) in the paper Orlov and Solovaya, 1974). $\overline{G}_2 = G_2/L_1$, $\overline{c} = c/L_1$, and m is the relation of mean motions. Denote

$$B_3' = \frac{6\,\delta\,B_3}{\overline{G}_2^2\,A_1}\,, \tag{17}$$

$$\kappa_1 \;=\; \left[1+\frac{1}{4}\,\frac{\gamma\,A_3\,m^2}{(1-e_2^2)^{\frac{3}{2}}}\right] n_1\,, \qquad \kappa_2 = \left[m-\frac{3}{16}\,\frac{\gamma\,A_3\,m^2}{\overline{G}_2\,(1-e_2^2)}\right] n_1\,,$$

$$\kappa_3 \;=\; \frac{3}{8}\,\frac{\gamma\,\delta\,m^2}{\overline{G}_2^2\,(1-e_2^2)^{\frac{3}{2}}}\,n_1\,, \qquad \kappa_4 = -\frac{3}{16}\,\frac{\gamma\,A_3\,m^2}{\overline{G}_2\,(1-e_2^2)^{\frac{3}{2}}}\,n_1\,, \tag{18}$$

where

$$\delta \;=\; \sqrt{(\xi_3-\xi_1)\,(\xi_4-\xi_1)\,(\xi_5-\xi_1)}\,, \qquad \gamma = m_2/\,(m_0+m_1+m_2)\,, \tag{19}$$

$$A_3 = \left(1 - 3\,q_0^2\right)\left(5 - 3\,\eta_0^2\right) - 15\,\left(1 - q_0^2\right)\left(1 - \eta_0^2\right)\cos\left(2\,g_1\right). \tag{20}$$

Equations (3)–(9) show that the functions contain terms linear with regard to the time t, terms linear with regard to u, and periodic terms with regard to u with period $2K$, where K is a complete elliptical integral. The dependence between u and the time t is defined by:

$$I_1(u) = B_3' + \kappa_3\,(t - t_0)\ . \tag{21}$$

The type of the orbit is defined by the sign of the $\overline{h}$ value. $\overline{h} = A_3 - A_{3cr}$, where A_{3cr} is such a value of A_3 when the roots of polynomial in denominator ξ_1 and ξ_2 equals 1. If $\overline{h} < 0$, the orbit is circular, the argument of periastron has the secular motion.

If $\overline{h} > 0$, the argument of periastron librate periodically at about 90° or 270°.

Pick up the secular terms from equations (4)–(9). For $t = t_0$ the dependence between constant B_3' and the initial u is following:

$$B_3' = I_1(u_0)\,. \tag{22}$$

We rewrite the integral $I_1(u)$ as the sum of the linear term with regard to u plus the periodic part:

$$\int_0^u \frac{1}{\sqrt{\sigma}}\,\mathrm{d}u = \frac{u}{K}\,I_1(K) + \int_0^u \left(\frac{1}{\sqrt{\sigma}} - \frac{I_1(K)}{K}\right)\mathrm{d}u\,, \tag{23}$$

where

$$I_1(K) = \int_0^K \frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,, \tag{24}$$

$$\Sigma_1 = \frac{1}{K}\int_0^K \frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,. \tag{25}$$

The second term in the right side of equation (23) is the periodic part

$$\Phi_1(u) = \int_0^u \left(\frac{1}{\sqrt{\sigma}} - \frac{I_1\,(K)}{K}\right)\mathrm{d}u\,. \tag{26}$$

Then

$$I_1(u) = \Sigma_1 u + \Phi_1(u)\,. \tag{27}$$

The relation between u and the time t can be rewritten as

$$\Sigma_1\,u + \Phi_1\,(u) = I_1\,(u_0) + \kappa_3\,\left(t - t_0\right)\ , \tag{28}$$

when u increases to $4\,K$, the value g_1 increases by $2\,\pi$, the time t changes by the period P_3:

$$P_3 = \frac{4\,K\,\Sigma_1}{\kappa_3}\,. \tag{29}$$

Denoting the mean motion of periastron of the inner orbit as ν_3, which is relation $\dfrac{2\pi}{P_3}$, then we can write

$$g_1 = \nu_3\,(t - t_0) + \Phi_1(u)\,. \tag{30}$$

Analogously, denote

$$\begin{aligned}
\Sigma_2 &= \frac{1}{K}\int_0^K \frac{\mathrm{sn}^2 u}{\sqrt{\sigma}}\,\mathrm{d}u\,,\\
\Sigma_3 &= \frac{1}{K}\int_0^K \frac{\mathrm{sn}^2 u}{1 - b_1^2\,\mathrm{sn}^2 u}\,\frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,,\\
\Sigma_4 &= \frac{1}{K}\int_0^K \frac{\mathrm{sn}^2 u}{1 - b_2^2\,\mathrm{sn}^2 u}\,\frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,,\\
\Sigma_5 &= \frac{1}{K}\int_0^K \frac{\mathrm{sn}^2 u}{1 + b_3^2\,\mathrm{sn}^2 u}\,\frac{1}{\sqrt{\sigma}}\,\mathrm{d}u\,,
\end{aligned} \tag{31}$$

to get the next expression of the integrals

$$I_i\,(u) = \Sigma_i\,u + \varPhi_i\,(u)\ . \tag{32}$$

Denoting the mean motion ν_i, which equals $\dfrac{2\,\pi}{P_i}$, where P_i is the period of revolution, we find

$$\begin{aligned}
l_1 &= \nu_1\ (t - t_0) + \ \textit{periodic terms},\\
l_2 &= \nu_2\ (t - t_0) + \ \textit{periodic terms},\\
g_2 &= \nu_4\ (t - t_0) + \ \textit{periodic terms},\\
h_1 &= \nu_5\ (t - t_0) + \ \textit{periodic terms}\,,
\end{aligned} \tag{33}$$

where

$$\nu_1 = \kappa_1 + \frac{\kappa_3}{\Sigma_1}\,(\Sigma_1\,Q_1 + \Sigma_2\,Q_2 + \ \Sigma_3\,Q_3)\ , \tag{34}$$

$$\nu_2 = \kappa_2\,, \tag{35}$$

$$\nu_3 = \frac{\pi}{2K}\,\frac{\kappa_3}{\Sigma_1}\,, \tag{36}$$

$$\nu_4 = \kappa_4 + \frac{\kappa_3}{\Sigma_1}\left(\Sigma_1\,Q_4 + \Sigma_2\,Q_5 + \Sigma_4\,Q_6 + \Sigma_5\,Q_7\right)\,, \tag{37}$$

$$\nu_5 = \frac{\kappa_3}{\Sigma_1}\left(\Sigma_1\,Q_8 + \Sigma_4\,Q_6 - \Sigma_5\,Q_7\right)\,. \tag{38}$$

Substituting in (34)–(38) κ_i from eqs. (18) we get:
a) The mean motion of the star S_1

$$\begin{aligned}\nu_1 \;=\; &\left\{1 + \frac{1}{16}\,\frac{\gamma\,m^2}{(1-e_2^2)^{\frac{3}{2}}}\left[4\,A_3 + \frac{6\,\delta}{\Sigma_1\,\overline{G}_2^2}\,\times\right.\right.\\ &\left.\left.\times\left(\Sigma_1\,Q_1 + \Sigma_2\,Q_2 + \,\Sigma_3\,Q_3\right)\right]\right\}\,n_1\,. \end{aligned} \tag{39}$$

b) The mean motion of the star S_2

$$\nu_2 = \left[m - \frac{1}{16}\,\frac{\gamma\,m^2}{(1-e_2^2)^{\frac{3}{2}}}\,\frac{3\,A_3\,\sqrt{1-e_2^2}}{\overline{G}_2}\right]n_1\,. \tag{40}$$

c) The mean motion of the periastron of the inner orbit

$$\nu_3 = \frac{1}{16}\,\frac{\gamma\,m^2}{(1-e_2^2)^{\frac{3}{2}}}\,\frac{3\,\pi\,\delta}{K\,\Sigma_1\,\overline{G}_2^2}\,n_1\,. \tag{41}$$

d) The mean motion of the periastron of the outer orbit

$$\begin{aligned}\nu_4 \;=\; &\frac{1}{16}\,\frac{\gamma\,m^2}{(1-e_2^2)^{\frac{3}{2}}}\left[-\frac{3\,A_3}{\overline{G}_2} + \frac{6\,\delta}{\Sigma_1\,\overline{G}_2^2}\,\times\right.\\ &\left.\times\left(Q_4\,\Sigma_1 + Q_5\,\Sigma_2 + Q_6\,\Sigma_4 + Q_7\,\Sigma_5\right)\right]n_1\,. \end{aligned} \tag{42}$$

e) The mean motion of the node

$$\nu_5 = \frac{1}{16}\,\frac{\gamma\,m^2}{(1-e_2^2)^{\frac{3}{2}}}\,\frac{6\,\delta}{\Sigma_1\,\overline{G}_2^2}\,\left(Q_8\,\Sigma_1 + Q_6\,\Sigma_4 - Q_7\,\Sigma_5\right)\,n_1\,. \tag{43}$$

This solution allows to investigate the dynamical evolution of the stellar system over a long time interval, to calculate maximum and minimum values of the eccentricities and to consider the question about the stability of systems.

3. The application

Consider the case, in which for a stellar system all six Keplerian elements and the direction of radial velocities in the neighbourhood of the nodes are known. Orbits of stars in such a system are defined unambiguously. Using the formulas of the previous section we can explain the character of the evolution of orbits of the close pair and the distant component. It is possible to learn the secular perturbations, using formulas (41–43), and long-periodic perturbations, using formulas (4–9) perturbations, of the periastrons and the nodes. Moreover, the results of such investigation can be used for the determination of favourable conditions to receive the perturbations from observations.

As an example of such a case the real triple system ξ UMa was selected. For this system the osculating elements and masses are known from observations covering a 175-year interval. The osculating elements of the stars with the masses $m_0 = 0.83$, $m_1 = 0.30$, and $m_2 = 0.92$ on the epoch $T_0 = 1900.0$ was taken from the paper of Heintz (1967).

The elements of orbits:

the inner orbit:	the outer orbit:
$a_1 = 1.56$ AU	$a_2 = 19.46 AU$
$e_1 = 0.56$	$e_2 = 0.414$
$T_{\pi_1} = 1935.41$	$T_{\pi_2} = 1935.17$
$\omega_1 = 146.00^o$	$\omega_2 = 127.5^o$
$\Omega_1 = 326.00^o$	$\Omega_2 = 101.5^o$
$i_1 = 86.30^o$	$i_2 = 122.65^o$.

The long-period perturbations for the system ξ UMa calculated by the formulae of the theory and those obtained by numerical integrations were compared in the paper of Solovaya and Pittich (2001). These two sets of perturbations agree quite well. Therefore, the theory can be used for the study of the long-periodic orbital evolution of the triple stellar system.

The system ξ UMa is the only system for which the perturbations of the outer orbit were found from the observations. Heintz (1967) showed that after several revolutions the elements of the far orbit are changed and they are not constant. He calculated the perturbations in the inclination, in the argument of the periastron, and in the node by the numerical integration within the 100 year interval. But he have supposed that the orbit of the close pair is a Keplerian ellipse. Orlov and Solovaya (1986) compared Heintz's results with results which they received from the computations using formulae of their theory for the motion of the outer orbit. The results of the calculations are presented graphically for the interval from 1750 to 2125, with a step of 25 years. They received the coincide of orders. For a more accurate comparison it is necessary to carry out the processing of the observations of the outer orbit using the formulae of their theory.

But Heintz did not find the perturbation of the close pair of the system, maybe because its period is very small. In paper of Orlov and Solovaya (1982) the character of the evolution of the inner orbit of the system was investigated. These authors found some difference in the selected elements of the orbit of the close pair. In particular, the value of the argument of periastron differs on the 180 degree.

The application of the theory for obtaining the qualitative characteristics of the stellar system is shown in the following steps. One should start from the definition of the constant $\overline{h}$, the sign of which allow to do the conclusion about the type of the inner orbit. For the system ξ UMa the value $\overline{h}$ has the positive sign. It means that the periastron of the close pair librates around g_1 within the range $262.96^\circ < g_1 < 277.04^\circ$.

The period of the double frequency of the libration is $P_3 = 2568.5$ years. The mean motion of the periastron of the outer orbit $\nu_4 = -14.97^\circ\,\mathrm{century}^{-1}$. The mean motion of the node is $\nu_5 = 14.56^\circ\,\mathrm{century}^{-1}$.

We compared the result for the mean motion of the node with those obtained by Brown (1936) with the formulae of his theory, 3° in 30 years. The discrepancy between these two results can be explained by the difference in the orbit of the distant component. In Brown's theory it is supposed that the orbit of the distant component is a fixed Keplerian ellipse. In our theory the orbit of the distant component is perturbed.

The semi-major axes of ξ UMa do not have the secular perturbations. The eccentricity of the close pair undergoes periodical variations from $e_{1_{min}} = 0.529$ to $e_{1_{max}} = 0.632$. The period of the variation is about 1334 years. The mutual inclination is subject to variations from $J_{min} = 128.50^\circ$ to $J_{max} = 132.35^\circ$. Under the influence of the distant component the motion at the inner orbit is accelerated to the value $\Delta\nu_1 = 0.129^\circ\,\mathrm{year}^{-1}$ and the motion of the distant component decreases to the value $\Delta\nu_2 = -0.0040^\circ\,\mathrm{year}^{-1}$. The both orbits are stable.

If we take into account the third term in the Hamiltonian, which depends on two angular variables, on the argument of periastron of the inner orbit (close pair) g_1 and the argument of periastron of the outer orbit (distant component) g_2, then the eccentricity of the outer orbit has long-periodic perturbations (Solovaya and Pittich, 2002).

The second case, when the inclinations of orbits for stellar systems in catalogues are given with the double sign, for the direction of radial velocities are not known, is more complicated. The theory allows to get the precise sign of the inclination, because for one sign the system is unstable.

We will demonstrate this on an example of the stellar system ζ Aqu (ADS15971). The close pair has the period of revolution 25.5 years, while the distant component has the period of revolution around the centre of mass of the close pair 600 years. The elements of orbits were taken from Frantz's work (1958). All angular elements were adjusted to the epoch of 2000 year.

The elements of orbits:

the inner orbit: the outer orbit:

$$\begin{aligned} a_1 &= 9.02\ \mathrm{AU} & a_2 &= 93.35 AU \\ e_1 &= 0.20 & e_2 &= 0.45 \\ T_{\pi_1} &= 1954.50 & T_{\pi_2} &= 1972.75 \\ \omega_1 &= 195.40^\circ & \omega_2 &= 73.62^\circ \\ \Omega_1 &= 177.30^\circ & \Omega_2 &= 124.37^\circ \\ i_1 &= \pm 57.80^\circ & i_2 &= \pm 137.00^\circ . \end{aligned}$$

Consider two variants of the evolutionary characteristic which correspond to the combinations of signs of the inner and outer orbit:
a) $i_1 = +57.80°$, $i_2 = -137.00°$,
b) $i_1 = +57.80°$, $i_2 = +137.00°$.

The total angular momentum of the system will be:
a) $c = 43.3675\ \mathrm{AU}^2\,\mathrm{M}_\odot\,\mathrm{century}^{-1}$,
b) $c = 46.3646\ \mathrm{AU}^2\,\mathrm{M}_\odot\,\mathrm{century}^{-1}$.

In the case a) the minimum and the maximum values of the eccentricity of the inner orbit change within the range $e_{1_{min}} = 0.1599$ and $e_{1_{max}} = 0.4578$.

It is evident that with such a combination of the inclination a close approach is not possible. The constant $\bar{h}$, which defines the type of the orbit, is negative and equals -0.297046. It means that the periastron of the orbit of a close pair has the secular motion. The mean motion ν_3 can be defined by formula (41). It is equal to $1.33°\,\mathrm{century}^{-1}$. The mean motion of the periastron of the outer orbit ν_4 is defined by formula (42) and is equal to $-1.35°\,\mathrm{century}^{-1}$. The motion of the node ν_5 is defined by formula (43). It is equal to $1.40°\mathrm{century}^{-1}$. The motion of the periastron of the outer orbit and that of the node go in the opposite directions. The period of change of the angular elements is about 30 000 years.

When the components are moving on unperturbed orbits, they have the following mean motions:

$$n_1 = k\sqrt{\frac{m_0 + m_1}{a_1^3}}, \qquad n_2 = k\sqrt{\frac{m_0 + m_1 + m_2}{a_2^3}}\,. \tag{44}$$

However, they are moving on perturbing orbits and the mean motion of a close pair slows down under the influence of a distant star to the value

$$\Delta\nu_1 = \nu_1 - n_1 = -0.900^\circ\,\mathrm{century}^{-1}\,. \tag{45}$$

The mean motion of a distant star grows under the influence of a close pair to the value

$$\Delta\nu_2 = \nu_2 - n_2 = +0.037^\circ\,\mathrm{century}^{-1}\,. \tag{46}$$

The mutual inclination of the orbits is $J = 137.09^\circ$.

We will call the angle of mutual inclination peculiar when $\overline{c} = \overline{G}_2$. The value of the cosine of the peculiar angle will be defined by the formula:

$$q_p = -\frac{\eta}{2\,\overline{G}_2}\,, \tag{47}$$

where $\eta = \sqrt{1-e_1^2}$. If the outer orbit is unperturbed, then the peculiar angle equals to 90°. The results of the investigation showed that if in the initial moment the angle of mutual inclination of the orbits is close to the peculiar angle, then in the process of evolution the system will reach the state when the eccentricity of the inner orbit will be close to 1 irrespective of the initial value of the eccentricity. The peculiar angle for the variant a) is 92.58°.

The system has different characteristics of dynamical evolution in case b). The mutual angle between orbits is 91.71°. The peculiar angle is 92.58°. The minimum and the maximum values of the eccentricity of inner orbit will be between the limits $e_{1\,min} = 0.1093$ and $e_{1\,max} = 0.9998$. In this case the minimum distance between the components can be $r_\pi = 0.27 \times 10^6$ km. It is $0.4\,R_\odot$. (The radius of the Sun, $R_\odot$, is 696 000 km). The period of the change of the eccentricity is about 11 000 years. This approach can lead to tidal phenomena. The formulae of the theory allow to investigate the characteristics of such approaches, their periodicity, the time when the periastron distance has the minimum value, the interval of the time in which the periastron distance cannot exceed a given limit.

The periastron distance of the inner orbit is defined by the following formula:

$$r_\pi = a_1\,(1 - e_1)\,, \tag{48}$$

or

$$r_\pi = a_1\,(1 - \sqrt{1-\xi})\,, \tag{49}$$

where

$$\xi = \xi_1 + (\xi_2 - \xi_1)\,\mathrm{sn}^2 u\,. \tag{50}$$

The minimum distance will be achieved in the moment of the time t, when the argument u has the value $0,\ 2K, \ldots$, where K is the complete elliptical integral,

$$K = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2\,\sin^2\varphi}}\,. \tag{51}$$

Define the interval of time over which the periastron distance will have the minimum value. Rewrite formula (28) as:

$$I_1(2\,K) = I_1(u_0) + \kappa_3(t - t_0)\,, \tag{52}$$

taking into account that $I_1(2\,K) = 2\,I_1(K)$ and find

$$t_1 - t_0 = \frac{1}{\kappa_3}(2\,K\,\Sigma_1 - I_1(u_0))\,. \tag{53}$$

Thus, the moment t_1 is the closest to t_0, supposing that $t_1 > t_0$, and the value r_π attains its minimum value

$$r_{\pi(min)} = a_1\left(1 - \sqrt{1 - \xi_1}\right)\,. \tag{54}$$

Now, we may establish for how long the periastron distance r_π after achieving its minimum value will not exceed the given value ρ, i.e.

$$r_\pi \leq \rho\,. \tag{55}$$

Substituting the left side of (55) by (54), and using (50), we may obtain the inequality:

$$\mathrm{sn}^2 u \leq \frac{1}{\xi_2 - \xi_1}\left[\frac{\rho}{a_1}\left(2 - \frac{\rho}{a_1}\right) - \xi_1\right]\,. \tag{56}$$

The limited value of u under condition (55) is:

$$\begin{aligned} \mathrm{sn} u \;&=\; +\sqrt{\frac{1}{\xi_2 - \xi_1}\left[\frac{\rho}{a_1}\left(2 - \frac{\rho}{a_1}\right) - \xi_1\right]}\,, \\ cnu \;&>\; 0\,. \end{aligned} \tag{57}$$

Denote u as u_ρ when $t = t_\rho$, for the argument $u \geq 2\,K$. u_ρ can be defined by the following integral:

$$I_1(u_\rho) = \int_0^{u_\rho} \frac{\mathrm{d}u}{\sqrt{1 - 2\,\beta\,\varepsilon\,\mathrm{sn}^2 u + \varepsilon^2\,\mathrm{sn}^4 u}}\,. \tag{58}$$

The condition (55) will be valid in the interval

$$t_\rho - t_1 = \frac{1}{\kappa_3}\left[I_1(u_\rho) - 2\,K\,\Sigma_1\right]\,. \tag{59}$$

The third application deals with close binary systems, supposing the existence of a far third component. For such a system all orbital elements and the mass of the third component cannot be obtained from observations. Only the orbital period and the light equation may be known for the third component from observational data. The theory allows to get the orbital elements of the supposed component so that the close binary system should stay stable. For the illustration the system AS Cam was considered. Its orbital elements were taken from the paper of Kolosova (2001):

$$\begin{array}{ll}
m_0{=}3.3\mathrm{M}_\odot\,, & m_1 = 2.5\ \mathrm{M}_\odot\,, \\
a_1{=}0.08 \pm 0.003\ \mathrm{AU}\,, & e_1 = 0.1705 \pm 0.0015\,, \\
P_1{=}3.43\ \mathrm{d}\,, & i_1 = 88.78^\mathrm{o}\,, \\
\omega_1{=}57.14^\mathrm{o}\,, & \Omega_1 = 0^\mathrm{o}\,, \\
R_1{=}2.53\mathrm{R}_\odot\,, & R_2 = 1.90\ \mathrm{R}_\odot\,,
\end{array}$$

where R_1 and R_2 are the radii of the components and P_1 the period of the revolution of the binary system.

We know only the following data of the third component of the system:

$$\frac{a_2\,\sin i_2}{c} = 4.18\ \mathrm{min}\,, \qquad e_2 = 0.5\,, \qquad P_2 = 805\ \mathrm{d}\,, \tag{60}$$

where the first equation is the light equation and c is the velocity of light. The motion of the apsid's line of this binary system shows the presence of a distant component with the mass of the order of the mass of the Sun. Kolosova (2001), using the previous intermediate orbit and data about the third component, defined the supposed parameters of the orbit and the mass by which the binary system can exist over a long time. The possible angle of the inclination was defined from the light equation for two supposed masses of the distant component: a) $m_2 = 1$, b) $m_2 = 1.5$ (mass in solar mass unit $M_\odot$). The motion of the distant component can be prograde ($i_2 = 9.0°$) and retrograde ($i_2 = 171.0°$). The node of the component varied from 0° to 360°. The condition of stability is $r_\pi > R_1 + R_2$. Then the critical value of the eccentricity will be $e_{cr} < 0.74$. It turned out that the results are not sensitive to the mass of the distant component.

The calculation showed that when the motion of the distant component is prograde, the eccentricity of the binary system does not exceed the critical value $e_{max} = 0.61$ for a) variant, and $e_{max} = 0.64$ for b) variant. In this case large tidal phenomena cannot occur. When the motion of the distant component is retrograde, the eccentricity of the binary system will exceed the critical value, for $\Omega = 0°$ through 900 years, for $\Omega = 180°$ through 1400 years. The binary system will be destroyed.

4. Conclusion

We have derived the application of the analytical theory of the general three-body problem to real triple star systems ξ UMa, ζ Aqu, and AS Cam. The systems differ from each other by their sets of orbital elements. In the first case the orbits are defined from observations unambiguously. In the second case the longitude of the nodes of both orbits are estimated from observations not unambiguously. In the third case the distant component lacks the full set of elements.

We showed that the theory can be used to investigate the dynamic evolution of triple stellar systems on the cosmogonic time interval. It allows to define the secular and long-periodic perturbations of the periastrons of orbits and the precession of its orbital plane. If the inclination of the orbits are given in the catalogue with a double sign, the theory indicates the precise sign of the inclination. Also the theory allows to establish the moments of the close approaches of components of the inner pair and the duration of these close approaches. For close binary systems, in which the existence of a distant third component is supposed, the theory allows to precise its orbit. The theory can be applied to extraplanets for establishment their orbital stability and the definition of some insufficient orbital elements.

Acknowledgements. This work was supported by the Slovak Academy of Sciences Grant VEGA No. 2/1005/21. The computations were done on the ORIGIN 2000 computer of the Computer center of the Slovak Academy of Sciences.

References

Brown, E.: 1936, *Mon. Not. R. Astron. Soc.* **97**, 116
Frantz, O.: 1958, *Astron. J.* **63**, 325
Heintz, W.D.: 1967, *Astron. Nachr.* **289**, 269
Kolosova, E.A.: 2001, *Vestn. Moskov. Univ.* **1**, 10
Orlov, A.A., Solovaya, N.A.: 1974, *Trudy GAISH* **XLV**, 119
Orlov, A.A., Solovaya, N.A.: 1982, *Vestn. Mosk. U. Fiz AS* **23**, 21
Orlov, A.A., Solovaya, N.A.: 1986, *Vestn. Mosk. U. Fiz AS* **27**, 74
Orlov, A.A., Solovaya, N.A.: 1988, in *The Few Body Problem*, ed.: M.J. Valtonen, Kluwer Acad. Publish., Dordrecht, 243
Solovaya, N.A., Pittich, E.M.: 2001, in *New Developments in the Dynamics of Planetary Systems*, eds.: R. Dvorak and J. Henrard, Kluwer Acad. Publish., Dordrecht, 319
Solovaya, N.A., Pittich, E.M.: 2002, *Contrib. Astron. Obs. Skalnaté Pleso* **32**, 117