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Homotopy continuation method for solving hyperbolic form of Kepler's equation

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Abstract. In this paper, error formulae for solving a hyperbolic form of Kepler's equation using the homotopy continuation method will be established for an arbitrary order of convergence $m \geq 2$.

Key words: homotopy –Kepler's equation – celestial mechanics

1. Introduction

Many instances of hyperbolic orbits occur in the solar system and, recently, among the artificial satellites, lunar and solar problems. Moreover, in some cases of the orbit determination for an elliptic orbit it may very well happen (Escobal, 1975) that during the solution process (usually iteration), the eccentricity becomes greater than unity and the orbit becomes hyperbolic. In addition, in interplanetary transfer, the escape from the departure planet and the capture by the target planet involves hyperbolic orbits (Gurzadyan, 1996). On the other hand, in the orbit determination of visual binaries provisional hyperbolic orbits are used to represent the perienteron section of high - eccentricity orbits of a long and indeterminate period (Knudsen, 1953). In fact, we should handle hyperbolic orbits frequently when integrating a perturbed motion with the initial condition of nearly parabolic orbits (Fukushima, 1997).

From the above it is then clear that the hyperbolic orbits not only exist naturally, but can also be used to solve some critical orbital situations. The position - time relation in hyperbolic orbits is known as Kepler's equation for the hyperbolic case and is given as

$$M = e \sinh G - G; \quad e > 1; \quad M \geq 0, \tag{1}$$

where G is the eccentric anomaly for a hyperbolic orbit, and M is the mean anomaly (Danby, 1988). Equation (1) is transcendental and usually solved by iterative methods, which in turn need: (a) initial guess and (b) an iterative scheme. In fact, these two points are not separated from each other, but there is a full agreement that even accurate iterative schemes are extremely sensitive to initial guess. Moreover, in many cases the initial guess may lead to a drastic situation between divergent and very slow convergent solutions.

In the field of numerical analysis, very powerful techniques have been devoted (Allgower and George, 1993) to solve transcendental equations without any a priori knowledge of the initial guess. These techniques are known as homotopy continuation methods. These methods were first applied to the deterministic orbit determination in NASA (Vallado, 1997). In addition, the method was employed at the Goddard Space Flight Center to support the preliminary orbit determination using tracking data from both tracking and data Relay satellite system and from traditional ground-based tracking stations (Montenbruck and Gill, 2000). Very recently (Sharaf and Sharaf, 2003), the method was for the first time applied to the universal initial value problem of space dynamics. The present paper is devoted to complete what was started in (Sharaf and Sharaf, 2003) by establishing error formulae for solving the hyperbolic form of Kepler's equation using the homotopy continuation method of an arbitrary order of convergence $m \geq 2$.

2. Development

Error formulae for solving the hyperbolic form of Kepler's equation using the homotopy continuation method of an arbitrary order of convergence $m \geq 2$ will be developed in what follows.

2.1. Homotopy continuation method for solving $Y(x) = 0$

Suppose one wishes to obtain a solution of a single non-linear equation in one variable x (say)

$$Y(x) = 0, \tag{2}$$

where $Y : R \rightarrow R$ is a mapping which for our application is assumed to be smooth, that is the map has as many continuous derivatives as required. Let us consider the situation in which no a priori knowledge concerning the zero point of Y is available. Since we assume that such *apriori* know-ledge is not available, then any of the iterative methods will often fail to calculate the zero $\overline{x}$, because poor starting value is likely to be chosen. As a possible remedy, one defines a homotopy or deformation $H : R \times R \rightarrow R$ such that

$$H(x, 1) = Q(x); \quad H(x, 0) = Y(x),$$

where $Q : R \rightarrow R$ is a (trivial) smooth map having a known zero point and H is also smooth. Typically, one may choose a convex

$$H(x, \lambda) = \lambda Q(x) + (1 - \lambda) Y(x), \tag{3}$$

and attempt to trace an implicitly defined curve $\Phi(z) \in H^{-1}(0)$ from a starting point $(x_1, 1)$ to a solution point $(\overline{x}, 0)$. If this succeeds, then a zero point $\overline{x}$ of Y is obtained.

The curve $\Phi(z) \in H^{-1}(0)$ can be traced numerically if it is parameterized with respect to the parameter λ, then the classical embedding methods can be applied (Allgower and George, 1993).

2.2. One-point iteration formulae

Let $Y(x) = 0$ such that $Y : R \to R$is a smooth map and has a solution $x = \xi$ (say). To construct iterative schemes for solving this equation, some basic definitions are to be recalled as follows:

1. The error in the k^{th} iterate is defined as
$$\varepsilon_k = \xi - x_k.$$
2. If the sequence $\{x_k\}$ converges to $x = \xi$, then
$$\lim_{k\to\infty} x_k = \xi \tag{4}$$
3. If there exists a real number $p \geq 1$ such that
$$\lim_{i\to\infty} \frac{|x_{i+1} - \xi|}{|x_i - \xi|^p} = \lim_{i\to\infty} \frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} = K \neq 0, \tag{5}$$
we say that, the iterative scheme is of order p at ξ. The constant K is called the asymptotic error constant. For $p = 1$, the convergence is linear; for $p = 2$, the convergence is quadratic; for $p = 3, 4, 5$ the convergence is cubic, quadric and quintic, respectively.
4. One - point iteration formulae are those which use information at only one point. Here, we shall consider only stationary one point iteration formulae which have the form
$$x_{i+1} = R(x_i); \quad i = 0, 1, 2, ... \tag{6}$$
5. The order of one point iteration formulae could be determine either from:

   (a) The Taylor series of the iteration function $R(x_n)$ about ξ e.g. (Ralston and Rabinowitz, 1978), or from

   (b) The Taylor series of the function $Y(x_{k+1})$ about x_k (Danby and Burkard, 1983).

   On the bases of the latter approach it is easy to form a class of iterative formulae containing members of all integral orders (Sharaf and Sharaf, 1998) to solve equation (2) as
$$x_{i+1} = x_i + \delta_{i,m+2}; \quad i = 0, 1.2, ...; \quad m = 0, 1, 2, ... \tag{7}$$

where

$$\delta_{i,m+2} = \frac{-Y_i}{\sum_{j=1}^{m+1}(\delta_{i,m+1})^{j-1}Y_i^{(j)}/j!}; \quad \delta_{i,1} = 1; \quad \forall\, i \geq 0, \tag{8}$$

$$Y_i^{(j)} \equiv \frac{d^j Y(x)}{dx^j}|_{x=x_i}; \quad Y_i \equiv Y_i^{(0)}. \tag{9}$$

The convergence order is $m+2$ as shown from the error formula

$$\varepsilon_{i+1} = \frac{-1}{(m+2)!}\frac{Y^{m+2}(\xi)}{Y_i^{(1)}(\xi_1)}\,\varepsilon_i^{m+2}, \tag{10}$$

where ξlies between x_{i+1} and x_i and ξ_1 between x_{i+1} and ξ.

3. Error formulae

Equation (2) for the continuation method with $Q = G-1$ is written as

$$H(G,\lambda) \equiv H = \lambda(G-1) + (1-\lambda)\{e\sinh G - G - M\}, \tag{11}$$

consequently, the derivative formulae for H are

$$\begin{aligned} H^{(1)} &= \lambda + (1-\lambda)\{e\cosh G - 1\}, \\ H^{(2r)} &= (1-\lambda)e\sinh G, \qquad r \geq 1, \\ H^{(2r+1)} &= (1-\lambda)e\cosh G. \end{aligned} \tag{12}$$

In what follows two cases of m should be considered separately.

– **First Case:** m **even** ≥ 2

From Equations (10) and (12) we get

$$|\varepsilon_{i+1}| = \frac{1}{m!}\,|\,\frac{(1-\lambda)e\sinh\xi}{\lambda+(1-\lambda)e\cosh G_i - 1}\,|\,\varepsilon_i^m$$

and since there exists $\xi < G_i$ then

$$|\varepsilon_{i+1}| \leq \frac{1}{m!}\,|\,\frac{e\sinh G_i}{e\cosh G_i - 1}\,|\,\varepsilon_i^m. \tag{13}$$

as $\sinh G \geq G$ and $e > 1$, then we get from equation (2)

$$0 \leq G \leq \frac{M}{e-1},$$

and therefore

$$\max\{\frac{e\sinh G}{e\cosh G-1}\}=\frac{\sinh(\alpha/\beta)}{\beta},$$

where $\alpha = M/e$ and $\beta = 1-1/e$, hence

$$\frac{e\sinh G}{e\cosh G-1}\leq\frac{\sinh(\alpha/\beta)}{\beta},$$

and inserting this inequality into (13) we get

$$|\varepsilon_{i+1}|\leq\frac{1}{m!}\mid\frac{\sinh(\alpha/\beta)}{\beta}\mid\varepsilon_i^m;\qquad m\quad \text{even}\ \geq 2. \tag{14}$$

– **Second Case:** m **odd** > 1

As above we can show that

$$|\varepsilon_{i+1}|\leq\frac{1}{m!}\mid\frac{e\cosh G_i}{e\cosh G_i-1}\mid\varepsilon_i^m,$$

$$\frac{e\cosh G}{e\cosh G-1}\leq\frac{\cosh(\alpha/\beta)}{\beta},$$

thus

$$|\varepsilon_{i+1}|\leq\frac{1}{m!}\mid\frac{\cosh(\alpha/\beta)}{\beta}\mid\varepsilon_i^m;\qquad m\quad \text{odd}\ \geq 1. \tag{15}$$

Inequalities (14) and (15)are what we required to set up for error formulae for solving the hyperbolic form of Kepler's equation using the homotopy continuation method of arbitrary order of convergence $m \geq 2$.

4. Numerical test

- In order to demonstrate the importance of the homotopy method, let us compare the case $n = 2$ and $m = 10$ with the corresponding second order Newton iterative method with the same number of iterations m. The results are listed in Table I in which the first two columns feature the used eccentricity e and the mean anomaly M, the third and the fourth columns are the solutions of (1) using the homotopy and the Newton methods respectively, while the last two columns are the values of $e\sinh G - G - M$ for the respective solutions.
- Form this table (and also from many other tests),we see strong dependence of the accuracy of the Newton method on the initial guess(taken as $G_\circ = M$),

while the homotopy method gives very accurate solution for all values of M and e.

- This last result proves that the homotopy method does not need any a priori knowledge of the initial guess. This is the property which avoids the critical situations between divergent to very slow convergent solutions that may exist in other numerical methods which depend on the initial guess.

Table 1. Numerical test

e	M $(deg.)$	$G_{Homotopy}$ $(deg.)$	G_{Newton} $(deg.)$	$ErrorHomotopy$	$ErrorNewton$
5	2.5	0.583009	0.583009	-1.699×10^{-10}	0.0
10	5.0	0.527926	0.527926	-4.7037×10^{-11}	0.0
15	7.5	0.511502	0.511502	-1.55866×10^{-11}	3.44825×10^{-6}
20	10.0	0.50362	0.693142	-6.84608×10^{-12}	4.30673
25	12.5	0.498993	2.56071	-3.57758×10^{-12}	
30	15.0	0.495949	5.00539	-2.09432×10^{-12}	
35	17.5	0.493795	7.50047	-1.33227×10^{-12}	
40	20.0	0.49219	10.0	-8.98837×10^{-13}	
45	22.5	0.490949	12.5	-6.32383×10^{-13}	
50	25.0	0.489959	15.0	-4.68958×10^{-13}	

References

Allgower, E.L., George, K.: 1993, *Numerical Continuation Method*, Springer-Verlag, Berlin

Danby, J.M.A.: 1988, *Fundamentals of Celestial Mechanics*, 2nd Edition William - Bell, Inc. Richmond, Virginia, USA

Danby, J.M.A., Burkard, T.M.: 1983, *Celestial Mechanics* **31**, 95

Escobal, P.R.: 1975, *Methods of Orbit Determination*, Wiley, New York

Montenbruck, O., Gill, E.: 2000, *Satellite Orbits, Models-Methods-Applications*, Springer-Verlag, Berlin

Fukushima, T.: 1997, *Astron. J.* **113**, 1920

Gurzadyan, G.A.: 1996, *Theory of Interplanetary Flights*, Gorden, Breach Publishers, Australia, Canada, United Kingdom

Knudsen, N.W.: 1953, *Publ. Lund Obs.* **12**,

Ralston, A., Rabinowitz, P.: 1978, *A First Course In Numerical Analysis*, McGraw-Hill Kogakusha, Ltd, Tokyo

Sharaf, M.A.and Sharaf, A.A.: 1998, *Celestial Mechanics and Dynamical Astronomy* **69**, 331

Sharaf, A.M., Sharaf, A.A.: 2003, *Celestial Mechanics and Dynamical Astronomy* **86**, 351

Vallado, D.: 1997, *Fundamentals of Astrodynamics and Applications*, The McGraw Hill Companies, Inc., UAS